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Interpolation on the Torus using sk-Splines with Number Theoretic Knots

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For a fixed, continuous, periodic kernel K, an sk-spline is a function of the form $\mathrm{sk}(x)=c_0+\sum_{i=1}^{n} c_i K(x-x_i)$. In this paper we consider a generalization of the univariate sk-spline to the d-dimensional torus ($d \geqslant 2$), and give almost optimal error estimates of the same order, in power scale, as best trigonometric approximation on Sobolev's classes in L_q. An important component of our method is that the interpolation nodes are generated using number theoretic ideas.

1. INTRODUCTION

Let the d-dimensional torus $\mathbb{T}^d=\mathbb{R}^d/2\pi\mathbb{Z}^d$ and $\Delta_n=\{\mathbf{w}_1, ..., \mathbf{w}_n\} \subset \mathbb{T}^d$, be interpolation nodes. Then, if $K \in \mathbf{C}(\mathbb{T}^d)$ has mean value 0, an sk-spline on $\mathbb{T}^d$ is a function of the form

$$\mathrm{sk}(\mathbf{w})=c_0+\sum_{i=1}^{n} c_k K(\mathbf{w}-\mathbf{w}_i),$$

where

$$\sum_{i=1}^{n} c_i=0.$$

For a continuous function f the sk-spline interpolant to f on the grid Δ_n is denoted $\mathrm{sk}(f, \Delta_n)$. Such functions are natural generalization of periodic polynomial splines, realized when K is a Bernoulli monospline of appropriate order. The sk-splines were introduced, and their basic theory developed, by Kushpel [6, 7]. In this paper we continue with the development of error estimates for sk-spline interpolation, begun, in the univariate

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case, in [8], and extended into higher dimensions in [4, 11]. For an overview of approximation by sk-splines see [9].

The above construction of an sk-spline using translates of a fixed kernel is identical to the construction of interpolants using a fixed conditionally positive definite radial function of order 1. This method of producing interpolants has received a lot of attention recently both for approximation in Euclidean space ([13, 14, 15]) and on the sphere ([2, 12, 16]). In fact, due to the compactness of the torus and the sphere, the notions of positive definiteness and conditional positive definiteness are almost identical: Each conditionally positive definite kernel gives rise to a positive definite functions by the addition of an appropriate polynomial. For the case of sk-splines this polynomial is simply a constant. The degree of conditional positive definiteness gives a lower bound for the degree of polynomial reproduction of the space of splines. Thus, sk-splines reproduce constants.

In the univariate case the optimal rate of convergence of spline interpolants with n knots and n points of interpolation on Sobolev's classes $W_p^r(\mathbb{T}^1)$ in $L_q(\mathbb{T}^1)$, $1 \leqslant p,\ q \leqslant \infty$, has order $n^{-r+(1/p-1/q)_+}$ as $n \to \infty$, for $r \in \mathbb{N}$, where $(a)_+ := \max\{a, 0\}$; see, e.g., [8]. The rate of best approximation from T_n, the subspace of trigonometric polynomials of degree $\leqslant n$, has the same order of convergence.

A fundamental question for multidimensional spline theory is whether the subspace of multidimensional splines (interpolants) will be as good as the subspace of trigonometric polynomials (of the same dimension) in the sense that they have the same rate of convergence on Sobolev's classes? If yes, then how do we construct optimal interpolants.

The purpose of the current article is to construct subspace of *sk*-splines which gives the same (up to some logarithmic factor) rate of convergence as the subspace of trigonometric polynomials (of the same dimension) on Sobolev's classes.

In the multidimensional setting, on $\mathbb{T}^d$, we consider the usual anisotropic Sobolev class $W_p^{\mathbf{a}}(\mathbb{T}^d) = W_p^{\mathbf{a}}$ with smoothness $\mathbf{a} = (a_1, ..., a_d)$, $0 < r = a_1 = \cdots = a_\nu < a_{\nu+1} \leqslant \cdots \leqslant a_d$. The order of approximation of functions from $W_p^{\mathbf{a}}$, in L_q, by trigonometric polynomials from the optimal hyperbolic cross containing $n(\log n)^{\nu-1}$ harmonics, is $(n^{-1}(\log n)^{\nu-1})^{r-(1/p-1/q)_+}$, $1 < p,\ q < \infty$ (see, e.g., [3]).

Interpolation at n points of functions $f \in W_p^{\mathbf{a}}$ by splines with gridded data gives the order of convergence $n^{(r- +1/p-1/q)/d}$, for $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$, $1/p - 1/q \geqslant 1/2$ (see [10, 11]) which is much slower than the order of best trigonometric approximation. In [1] the rate of convergence of Hermite interpolants, for $p = 2$, $q = \infty$, is explored. In the case of Lagrange interpolation they obtain the order of convergence $n^{-(r+1/2)/d}$.

Applying the apparatus of sk-splines we construct interpolants which have the same order of convergence, in the power scale, as best trigonometric approximation. In [4] we give such a result. Using more sophisticated techniques we can improve this result by decreasing the order of the logarithmic term in the error estimate.

An important component of our method is that the interpolation nodes are generated using prime numbers and associated number theoretic ideas. For a fixed prime number P let $\mathbb{Z}^d \supset G_p = \{\mathbf{g} = (g_1, ..., g_d): 1 \leqslant g_i \leqslant P-1,\ i = 1, ..., d\}$. Then for any fixed $\mathbf{g} \in G_P$, let $\Delta_P^{\mathbf{g}} = \{\mathbf{w}_j = 2\pi j \mathbf{g}/P: j = 0, 1, ..., P-1\} \subseteq \mathbb{T}^d$. For future reference we note that $-\mathbf{w}_j = \mathbf{w}_{P-j}$, $j = 0, 1, ..., P-1$, setting $\mathbf{w}_P = \mathbf{w}_0$.

Define the Fourier coefficients

$$\hat{K}(\mathbf{z}) = \int K(\mathbf{w})\, e^{-i\mathbf{wz}}\, d\mathbf{w}, \qquad \mathbf{z} \in \mathbb{Z}^d,$$

where $\mathbf{wz}$ denotes the scalar product of the vectors $\mathbf{w}$ and $\mathbf{z}$, and $d\mathbf{w}$ is the normalized Haar measure on the torus. We will consider kernels K of the form

$$K(\mathbf{w}) = \sum_{\mathbf{z} \in \mathbb{Z}^d/\{\mathbf{0}\}} \hat{K}(\mathbf{z})\, e^{i\mathbf{wz}}, \qquad \sum_{\mathbf{z} \in \mathbb{Z}^d/\{\mathbf{0}\}} \hat{K}(\mathbf{z}) < \infty,$$

where $\hat{K}(\mathbf{z}) > 0$, for all $\mathbf{z} \in \mathbb{Z}^d/\{\mathbf{0}\}$, that is, we assume that K has mean value zero and positive summable Fourier coefficients. Then, it is guaranteed (see Remark 1) that the interpolation problem

$$\mathrm{sk}(\mathbf{w}_i) = c_0 + \sum_{j=1}^{P} c_j K(\mathbf{w}_i - \mathbf{w}_j) = v_i, \qquad i = 1, ..., P,$$

$$\sum_{j=1}^{P} c_j = 0,$$

has a unique solution.

In Section 2 we give an explicit representation of the cardinal sk-spline on the knot set $\Delta_P^{\mathbf{g}}$, and give some useful facts concerning these splines. In Section 3 we give asymptotic error bounds for sk-spline interpolation, on $\Delta_P^{\mathbf{g}}$, of functions in Sobolev classes. Note that all equivalences in this paper are modulo P, and that all integrals are over the torus with respect to normalized Haar measure. Furthermore, the C will be used to denote a constant, not necessarily the same at each occurrence, which may depend on any parameter but P.

2. CARDINAL sk-SPLINES AND INTERPOLATING FOURIER MODES

The main tool for constructing cardinal sk-splines is the discrete Fourier transform on the knot sequence $\Delta_P^{\mathbf{g}}$. Therefore, the proofs of many of the results in this section are tractable to standard direct calculation. With this in mind we leave out all proofs except that of Theorem 2.

For $\lambda = 1, ..., P-1$, let

$$\tilde{K}_\lambda(\mathbf{w}) = \sum_{s=0}^{P-1} K(\mathbf{w} + \mathbf{w}_s)\, e^{-2\pi i\lambda s/P}.$$

LEMMA 1.

$$\tilde{K}_\lambda(\mathbf{w} - \mathbf{w}_s) = e^{-2\pi i\lambda s/P} \tilde{K}_\lambda(\mathbf{w}).$$

THEOREM 1. *Suppose that* $\tilde{K}_\lambda(\mathbf{0}) \neq 0$, $\lambda = 1, 2, ..., P-1$. *Then*

$$\widetilde{\mathrm{sk}}_P^{\mathbf{g}}(\mathbf{w}) := \frac{1}{P}\left(1 + \sum_{\lambda=1}^{P-1} \frac{\tilde{K}_\lambda(\mathbf{w})}{\tilde{K}_\lambda(\mathbf{0})}\right)$$

satisfies

$$\widetilde{\mathrm{sk}}_P^{\mathbf{g}}(\mathbf{w}) = \begin{cases} 1, & \mathbf{w} = \mathbf{0}, \\ 0, & \mathbf{w} = \mathbf{w}_j, \quad 1 \leqslant j \leqslant P-1. \end{cases}$$

In other words, $\widetilde{\mathrm{sk}}_P^{\mathbf{g}}$ *is a Lagrange function for the points* $\mathbf{w}_0, ..., \mathbf{w}_{P-1}$.

Remark 1. It is straightforward to show that $\widetilde{\mathrm{sk}}_P^{\mathbf{g}}$ is an sk-spline. Also, if $\hat{K}(\mathbf{z}) > 0$ for all $\mathbf{z} \in \mathbb{Z}^d$ then, for $\lambda = 1, ..., P-1$,

$$\begin{aligned} \tilde{K}_\lambda(\mathbf{0}) &= \sum_{s=0}^{P-1} K(\mathbf{w}_s)\, e^{-2\pi i\lambda s/P} \\ &= \sum_{s=0}^{P-1} \sum_{\mathbf{z} \in \mathbb{Z}^d} \hat{K}(\mathbf{z})\, e^{i\mathbf{z}\mathbf{w}_s - 2\pi i\lambda s/P} \\ &= \sum_{\mathbf{z} \in \mathbb{Z}^d} \hat{K}(\mathbf{z}) \sum_{s=0}^{P-1} e^{(2\pi i s/P)(\mathbf{z}\mathbf{g} - \lambda)} \\ &= P \sum_{\mathbf{z}\mathbf{g} \equiv \lambda} \hat{K}(\mathbf{z}) > 0, \end{aligned}$$

as the Fourier coefficients of K are all positive. Thus, as long as the Fourier series for K is absolutely convergent, sk-spline interpolation has a solution. We know that this is unique from Theorem 4 of [10].

LEMMA 2. *For all* $\mathbf{w} \in \mathbb{T}^d$,

$$\sum_{s=1}^{P} \widetilde{\mathrm{sk}}_P^{\mathbf{g}}(\mathbf{w} - \mathbf{w}_s) = 1.$$

LEMMA 3. (a) $\widetilde{\mathrm{sk}}_P^{\mathbf{g}}(-\mathbf{w}) = \widetilde{\mathrm{sk}}_P^{\mathbf{g}}(\mathbf{w})$.

(b) *If we write*

$$\widetilde{\mathrm{sk}}_P^{\mathbf{g}}(\mathbf{w}) = c_0 + \sum_{i=1}^{P} c_i K(\mathbf{w} - \mathbf{w}_i),$$

then

(i) $c_0 = 1/P$,

(ii) $c_{P-i} = c_i$, $i = 1, ..., P-1$.

For the following theorem we require some new notation. Let $\sum^*$ denote a sum in which the summation index ranges over all non zero elements of the summation set.

THEOREM 2. *Let* $E_{\mathbf{z}}(\mathbf{w}) = e^{i\mathbf{z}\mathbf{w}} - \sum_{j=0}^{P-1} e^{i\mathbf{z}\mathbf{w}_j} \widetilde{\mathrm{sk}}_p^{\mathbf{g}}(\mathbf{w} - \mathbf{w}_j)$. *Then*

$$|E_{\mathbf{z}}(\mathbf{w})| \leqslant 2 \begin{cases} 1, & \text{for any} \quad \mathbf{z} \in \mathbb{Z}^d, \\ S_{\mathbf{z}}/\hat{K}(\mathbf{z}), & \mathbf{z}\mathbf{g} \not\equiv 0, \end{cases}$$

where

$$S_{\mathbf{z}} = \sum_{\mathbf{g}\mathbf{k} \equiv 0}^{*} \hat{K}(\mathbf{k} + \mathbf{z}).$$

Proof. First let us set $n \equiv \mathbf{z}\mathbf{g}$, $0 \leqslant n < P$, so that $\mathbf{z}\mathbf{w}_j = 2\pi jn/P \pmod{2\pi}$. Then, by Lemma 1,

$$\begin{aligned} \sum_{j=0}^{P-1} e^{i\mathbf{z}\mathbf{w}_j} \widetilde{\mathrm{sk}}_P^{\mathbf{g}}(\mathbf{w} - \mathbf{w}_j) &= \sum_{j=0}^{P-1} e^{i\mathbf{z}\mathbf{w}_j} \left\{ \frac{1}{P} + \frac{1}{P} \sum_{k=1}^{P-1} \frac{\widetilde{K}_k(\mathbf{w} - \mathbf{w}_j)}{\widetilde{K}_k(\mathbf{0})} \right\} \\ &= \frac{1}{P} \sum_{j=0}^{P-1} e^{i\mathbf{z}\mathbf{w}_j} + \frac{1}{P} \sum_{k=1}^{P-1} \frac{\widetilde{K}_k(\mathbf{w})}{\widetilde{K}_k(\mathbf{0})} \sum_{j=0}^{P-1} e^{-2\pi i j(k-n)/P} \\ &= \begin{cases} 1, & n = 0, \\ \widetilde{K}_n(\mathbf{w})/\widetilde{K}_n(\mathbf{0}), & n \neq 0, \end{cases} \end{aligned} \tag{1}$$

using the discrete orthogonality of the complex exponentials. If $n=0$ the result is obvious. If $n \neq 0$ then

$$\begin{aligned}
\tilde{K}_n(\mathbf{w}) &= \sum_{j=0}^{P-1} K(\mathbf{w}+\mathbf{w}_j)\, e^{-2\pi i n j/P} \\
&= \sum_{j=0}^{P-1} \left\{ \sum_{k \in \mathbb{Z}^d} \hat{K}(\mathbf{k})\, e^{i\mathbf{k}(\mathbf{w}+\mathbf{w}_j)} \right\} e^{-2\pi i n j/P} \\
&= \sum_{\mathbf{k} \in \mathbb{Z}^d} \hat{K}(\mathbf{k})\, e^{i\mathbf{kw}} \sum_{j=0}^{P-1} e^{-2\pi i(n-\mathbf{kg})\, j/P} \\
&= P \sum_{\mathbf{kg} \equiv n} \hat{K}(\mathbf{k})\, e^{i\mathbf{kw}}.
\end{aligned}$$

Hence, if $n \neq 0$, using (1), we have

$$\begin{aligned}
|E_{\mathbf{z}}(\mathbf{w})| &= \left| e^{i\mathbf{zw}} - \frac{\sum_{\mathbf{kg} \equiv n} \hat{K}(\mathbf{k})\, e^{i\mathbf{kw}}}{\sum_{\mathbf{kg} \equiv n} \hat{K}(\mathbf{k})} \right| \\
&= \left| \frac{e^{i\mathbf{zw}} \sum_{\mathbf{kg} \equiv n} \hat{K}(\mathbf{k}) - \sum_{\mathbf{kg} \equiv n} \hat{K}(\mathbf{k})\, e^{i\mathbf{kw}}}{\sum_{\mathbf{kg} \equiv n} \hat{K}(\mathbf{k})} \right| \\
&\leqslant 2 \frac{\sum_{\mathbf{kg} \equiv n,\, \mathbf{k} \neq \mathbf{z}} \hat{K}(\mathbf{k})}{\sum_{\mathbf{kg} \equiv n} \hat{K}(\mathbf{k})} \\
&= 2 \frac{\sum^{*}_{\mathbf{kg} \equiv 0} \hat{K}(\mathbf{k}+\mathbf{z})}{\sum_{\mathbf{kg} \equiv 0} \hat{K}(\mathbf{k}+\mathbf{z})}.
\end{aligned}$$

The first part of the result follows easily and the second part because all of the Fourier coefficients of K are positive. ∎

3. ERROR ESTIMATES

In the following analysis we will require two simple inequalities. We will prove the first only, the second following in a similar fashion.

LEMMA 4. *Let* $\mathbf{t} = (t_1, ..., t_d)$ *and* $\mathbf{t}' = (t'_1, ..., t'_d)$ *satisfy* $t_i = t'_i = 1$, $i = 1, ..., \nu$, *and* $1 < t'_i < t_i$, $i = \nu+1, ..., d$ *and let* $\mathbf{s}$ *range over* $\mathbb{N}^d$. *Then, for* $\beta > 0$ *and* $n \geqslant 0$,

(a)

$$\sum_{\mathbf{ts} \leqslant n} 2^{\beta \mathbf{t}'\mathbf{s}} \leqslant C 2^{\beta n} n^{\nu - 1}, \tag{2}$$

(b)

$$\sum_{\mathbf{t}'\mathbf{s} \geqslant n} 2^{-\beta \mathbf{ts}} \leqslant C 2^{-\beta n} n^{\nu-1}. \tag{3}$$

Proof. (a) For $d=1$ the result is trivial, so we proceed by induction on d, distinguishing two cases; $\nu=1$ and $\nu>1$. If $\nu=1$ we write

$$\begin{aligned}\sum_{\mathbf{ts} \leqslant n} 2^{\beta \mathbf{t}'\mathbf{s}} &\leqslant \sum_{t_d s_d \leqslant n} 2^{\beta t'_d s_d} \sum_{t_1 s_1 + \cdots + t_{d-1} s_{d-1} \leqslant n - t_d s_d} 2^{\beta(t'_1 s_1 + \cdots + t'_{d-1} s_{d-1})} \\ &\leqslant C \sum_{t_d s_d \leqslant n} 2^{\beta t'_d s_d} 2^{\beta(n - t_d s_d)},\end{aligned}$$

using the inductive hypothesis. The last sum is bounded by a constant multiple of

$$2^{\beta n} \sum_{t_d s_d \leqslant n} 2^{-\beta s_d (t_d - t'_d)},$$

and the result for $\nu=1$ follows because $t_d > t'_d$, ensuring that the last sum is bounded independently of n. If $\nu>1$ we write

$$\begin{aligned}\sum_{\mathbf{ts} \leqslant n} 2^{\beta \mathbf{t}'\mathbf{s}} &\leqslant \sum_{t_1 s_1 \leqslant n} 2^{\beta t'_1 s_1} \sum_{t_2 s_2 + \cdots + t_d s_d \leqslant n - t_1 s_1} 2^{\beta(t'_2 s_2 + \cdots + t'_d s_d)} \\ &\leqslant C \sum_{t_1 s_1 \leqslant n} 2^{\beta t'_1 s_1} 2^{\beta(n - t_1 s_1)} (n - t_1 s_1)^{\nu-2} \\ &\leqslant C 2^{\beta n} n^{\nu-2} \sum_{t_1 s_1 \leqslant n} 2^{\beta s_1 (t'_1 - t_1)}\end{aligned}$$

again using the inductive hypothesis. However, $t_1 = t'_1 = 1$ as $\nu>1$, so the final sum is bounded by $n+1$ completing the proof for $\nu>1$. ∎

Before proceeding with our error estimate we fix the kernel $K_{\mathbf{a}}$ and define the anisotropic Sobolev class $W_p^{\mathbf{a}}$. Let the dimensions of the torus be ordered so that the multiindex $\mathbf{a} = (a_1, ..., a_d)$ satisfies $1 < r = a_1 = \cdots = a_\nu < a_{\nu+1} \leqslant \cdots \leqslant a_d$. Then, let

$$\hat{K}_{\mathbf{a}}(\mathbf{z}) = \underline{z}_1^{-a_1} \cdots \underline{z}_d^{-a_d},$$

where, for $n \in \mathbb{Z}$, $\underline{n} = \max\{|n|, 1\}$. Define

$$\|\phi\|_p = \begin{cases} \left\{\int |\phi(\mathbf{w})|^p \, d\mathbf{w}\right\}^{1/p} & 1 \leqslant p < \infty, \\ \text{ess sup } |\phi|, & p = \infty. \end{cases}$$

For $1 \leqslant p \leqslant \infty$ let $U_p = \{\phi : \|\phi\|_p \leqslant 1\}$. Then,

$$\begin{aligned} W_p^{\mathbf{a}} &= \{\psi : D^{\mathbf{a}}\psi \in U_p\} \\ &= \{c + K_{\mathbf{a}} * \phi : c \in \mathbb{R}, \phi \in U_p\}. \end{aligned}$$

For fixed $\mathbf{w} \in \mathbb{T}^d$, $\mathbf{g} \in G$, and $f \in W_p^{\mathbf{a}}$,

$$\begin{aligned} &|f(\mathbf{w}) - [\mathrm{sk}(f, \Delta_P^{\mathbf{g}})](\mathbf{w})| \\ &\quad = \left| c + (K_{\mathbf{a}} * \phi)(\mathbf{w}) - \sum_{j=1}^{P} [c + (K_{\mathbf{a}} * \phi)(\mathbf{w}_j)]\, \widetilde{\mathrm{sk}}_P^{\mathbf{g}}(\mathbf{w} - \mathbf{w}_j) \right| \\ &\quad = \left| (K_{\mathbf{a}} * \phi)(\mathbf{w}) - \sum_{j=1}^{P} (K_{\mathbf{a}} * \phi)(\mathbf{w}_j)\, \widetilde{\mathrm{sk}}_P^{\mathbf{g}}(\mathbf{w} - \mathbf{w}_j) \right|, \end{aligned}$$

for some $\phi \in U_p$, using Lemma 2.

For $\mathbf{z} \in \mathbb{Z}^d$, let us define the coefficients

$$c_{\mathbf{z}}(\mathbf{w}) = \int \left\{ K_{\mathbf{a}}(\mathbf{w} - \mathbf{y}) - \sum_{j=1}^{P} K_{\mathbf{a}}(\mathbf{w}_j - \mathbf{y})\, \widetilde{\mathrm{sk}}_P^{\mathbf{g}}(\mathbf{w} - \mathbf{w}_j) \right\} e^{-i\mathbf{z}\mathbf{y}}\, d\mathbf{y}.$$

Then,

$$c_{\mathbf{z}}(\mathbf{w}) = \hat{K}_{\mathbf{a}}(\mathbf{z})\, E_{\mathbf{z}}(\mathbf{w}), \tag{4}$$

where $E_{\mathbf{z}}(\mathbf{w})$ is defined in the statement of Theorem 2.

Using Hölder's inequality we can bound

$$\begin{aligned} &\left| K_{\mathbf{a}} * \phi(\mathbf{w}) - \sum_{j=1}^{P} (K_{\mathbf{a}} * \phi)(\mathbf{w}_j)\, \widetilde{\mathrm{sk}}_P^{\mathbf{g}}(\mathbf{w} - \mathbf{w}_j) \right| \\ &\qquad \leqslant \|\phi\|_p \left\| K_{\mathbf{a}}(\cdot - \mathbf{w}) - \sum_{j=1}^{P} K_{\mathbf{a}}(\mathbf{w}_j - \cdot)\, \widetilde{\mathrm{sk}}_P^{\mathbf{g}}(\mathbf{w} - \mathbf{w}_j) \right\|_{p'} \\ &\qquad \leqslant \left(\sum_{\mathbf{z} \in \mathbb{Z}^d} |c_{\mathbf{z}}(\mathbf{w})|^p \right)^{1/p}, \end{aligned} \tag{5}$$

using the Hausdorff–Young inequality for $1 \leqslant p \leqslant 2$ (see [17]).

To estimate the right hand side of (5) we divide $\mathbb{Z}^d$ into two regions, Γ_Q, including the origin, and its complement. The number Q depends on the prime number P. Let $\mathbf{b} = (p\mathbf{a} - \mathbf{e})/(pr - 1)$, where $\mathbf{e} = (1, ..., 1)$, and let $\mathbf{b}'$ be any multiindex satisfying $b_i' = 1$, $i = 1, ..., v$, and $1 < b_i' < b_i$, $i = v + 1, ..., d$. For any non-negative multiintegers $\mathbf{s}$, $\Psi_{\mathbf{s}} = \{\mathbf{z} \in \mathbb{Z}^d \text{: } 2^{s_i} - 1 \leqslant |z_i| < 2^{s_i + 1} - 1,$

$i=1, ..., d\}$. For fixed Q let m be the integer such that $2^{m-1} \leqslant Q < 2^m$. Then,

$$\Gamma_Q = \bigcup_{\mathbf{sb}' \leqslant m} \Psi_{\mathbf{s}}.$$

Remark 2. Γ_Q is approximately the set of multiintegers inside the hyperbolic cross $\hat{K}(\mathbf{z}) \leqslant 1/Q$.

PROPOSITION 1. *For every* $\mathbf{w} \in \mathbb{T}^d$,

$$\left(\sum_{\mathbf{z} \notin \Gamma_Q} |c_{\mathbf{z}}(\mathbf{w})|^p\right)^{1/p} \leqslant CQ^{-r+1/p}(\log Q)^{(\nu-1)/p}.$$

Proof. Using (4) and Theorem 2 we have

$$\begin{aligned}
\left(\sum_{\mathbf{z} \notin \Gamma_Q} |c_{\mathbf{z}}(\mathbf{w})|^p\right)^{1/p} &\leqslant 2\left(\sum_{\mathbf{z} \notin \Gamma_Q} (\hat{K}_{\mathbf{a}}(\mathbf{z}))^p\right)^{1/p} \\
&\leqslant 2\left(\sum_{\mathbf{sb}' \geqslant m} \sum_{\mathbf{z} \in \Psi} (\hat{K}_{\mathbf{a}}(\mathbf{z}))^p\right)^{1/p} \\
&\leqslant 2\left(\sum_{\mathbf{sb}' \geqslant m} \max_{\mathbf{z} \in \Psi_{\mathbf{s}}} (\hat{K}_{\mathbf{a}}(\mathbf{z}))^p \operatorname{card}(\Psi_{\mathbf{s}})\right)^{1/p} \\
&\ll \left(\sum_{\mathbf{sb}' \geqslant m} 2^{-p\mathbf{sa}} 2^{\mathbf{se}}\right)^{1/p},
\end{aligned}$$

since $\operatorname{card}\psi_{\mathbf{s}} \leqslant 2^d 2^{\mathbf{se}}$, and $\max_{\mathbf{z} \in \Psi_{\mathbf{s}}} \hat{K}_{\mathbf{a}}(\mathbf{z}) \leqslant C2^{-\mathbf{sa}}$. Thus

$$\begin{aligned}
\left(\sum_{\mathbf{z} \notin \Gamma_Q} |c_{\mathbf{z}}(\mathbf{w})|^p\right)^{1/p} &\leqslant \left(\sum_{\mathbf{sb}' \geqslant m} 2^{(-rp+1)(\mathbf{s}(p\mathbf{a}-\mathbf{e})/rp-1)}\right)^{1/p} \\
&\leqslant Cm^{(\nu-1)/p} 2^{-m(r-1/p)} \\
&\leqslant CQ^{-r+1/p}(\log Q)^{(\nu-1)/p},
\end{aligned}$$

because $2^{m-1} \leqslant Q$, where the penultimate step follows from (b) of Lemma 4. ∎

Bounding the sum

$$\left(\sum_{\mathbf{z} \in \Gamma_Q} |c_{\mathbf{z}}(\mathbf{w})|^p\right)^{1/p}$$

is a more delicate issue, and is the subject of the next, more involved, proposition. We remind the reader that $c_{\mathbf{z}}(\mathbf{w})$ depends on the particular choice of $\mathbf{g}$.

PROPOSITION 2. *Suppose $Q \leqslant [P/4]$. Then, there exists $\mathbf{g}^* \in G$ such that, for every $\mathbf{w} \in \mathbb{T}^d$,*

$$\left(\sum_{\mathbf{z} \in \Gamma_Q} |c_{\mathbf{z}}(\mathbf{w})|^p\right)^{1/p} \leqslant CP^{-r}(\log P)^{r\nu}\, Q^{1/p}(\log Q)^{(\nu-1)/p}.$$

Proof. First, using Theorem 2, we bound

$$\begin{aligned}\left(\sum_{\mathbf{z} \in \Gamma_Q} |c_{\mathbf{z}}(\mathbf{w})|^p\right)^{1/p} &\leqslant 2\left(\sum_{\mathbf{z} \in \Gamma_Q} S_{\mathbf{z}}^p\right)^{1/p} \\ &\leqslant 2(\mathrm{card}(\Gamma_Q))^{1/p} \max_{\mathbf{z} \in \Gamma_Q} S_{\mathbf{z}}. \qquad (6)\end{aligned}$$

Now,

$$\begin{aligned}\mathrm{card}(\Gamma_Q) &\leqslant 2^d \sum_{\mathbf{sb}' \leqslant m} 2^{\mathbf{es}} \\ &\ll 2^m m^{\nu-1}, \\ &\ll Q(\log Q)^{\nu-1}\end{aligned}$$

as $Q \leqslant 2^m$, where the first step follows by (a) of Lemma 4.

To prove the proposition we now need to show that there exists $\mathbf{g}^* \in G$ such that, for any $\mathbf{z} \in \Gamma_Q$,

$$\begin{aligned}S_{\mathbf{z}} &= \sum_{\mathbf{kg}^* \equiv 0}^{*} \hat{K}_{\mathbf{a}}(\mathbf{z}+\mathbf{k}) \\ &\leqslant CP^{-r}(\log P)^{r\nu}. \qquad (7)\end{aligned}$$

Let us put $\mathbf{k} = \mathbf{n}P + \mathbf{m}$, where $\mathbf{m} = (m_1, ..., m_d) \in \Omega_P := \{\mathbf{m} \in Z^d: -[(P-1)/2] \leqslant m_s \leqslant [P/2],\ 1 \leqslant s \leqslant d\}$. It is straightforward to show $|nP + m + z| \geqslant nm/4$, whenever $n \in \mathbb{Z}$, $|m| \leqslant P/2$, and $|z| \leqslant P/4$. Since $\mathbf{z} \in \Gamma_Q$ $\max_s |z_s| \leqslant P/4$ by assumption. It immediately follows that, for some constant C,

$$\hat{K}_{\mathbf{a}}(\mathbf{z}+\mathbf{k}) \leqslant C \begin{cases} \hat{K}_{\mathbf{a}}(\mathbf{n})\, \hat{K}_{\mathbf{a}}(\mathbf{m}), & \mathbf{n} \text{ and } \mathbf{m} \neq \mathbf{0}, \\ \hat{K}_{\mathbf{a}}([P\mathbf{n}/2]), & \mathbf{m} = \mathbf{0}. \end{cases}$$

Since $r>1$, the series for $S_{\mathbf{z}}$ is absolutely summable, and we can reorder the sum to give, in mind of the last inequality,

$$S_{\mathbf{z}} \leqslant C\left(\sum^{*} \hat{K}_{\mathbf{a}}(\mathbf{n}) \sum_{\mathbf{mg}\equiv 0,\, \mathbf{m}\in\Omega_P,}^{*} \hat{K}_{\mathbf{a}}(\mathbf{m}) + \sum^{*} \hat{K}_{\mathbf{a}}([P\mathbf{n}]/2)\right). \tag{8}$$

Direct calculation shows that the second sum on the right is bounded by CP^{-r} for some constant C. After summation over $\mathbf{n}\in\mathbb{Z}^d$ we see that

$$\begin{aligned}\sum^{*} \hat{K}_{\mathbf{a}}(\mathbf{n}) \sum_{\mathbf{mg}\equiv 0,\, \mathbf{m}\in\Omega_P,}^{*} \hat{K}_{\mathbf{a}}(\mathbf{m}) &\leqslant C \sum_{\mathbf{mg}\equiv 0,\, \mathbf{m}\in\Omega_P}^{*} \hat{K}_{\mathbf{a}}(\mathbf{m})\\ &\leqslant \left(\sum_{\mathbf{mg}\equiv 0,\, \mathbf{m}\in\Omega_P}^{*} (\hat{K}_{\mathbf{a}}(\mathbf{m}))^{1/r}\right)^{r},\end{aligned} \tag{9}$$

due to Minkowski's inequality.

Since $\mathbf{m}\neq\mathbf{0}$ then for the average

$$\begin{aligned}S &:= \frac{1}{(P-1)^d} \sum_{\mathbf{g}\in G} \sum_{\mathbf{mg}\equiv 0,\, \mathbf{m}\in\Omega_P}^{*} (\hat{K}_{\mathbf{a}}(\mathbf{m}))^{1/r}\\ &\leqslant P^{-1} \sum_{\mathbf{m}\in\Omega_P}^{*} (\hat{K}_{\mathbf{a}}(\mathbf{m}))^{1/r}\end{aligned}$$

because P is prime and, consequently, the maximum number of solutions of equation $\mathbf{mg}\equiv 0$ for any fixed $\mathbf{m}\in\Omega_P$ is $\leqslant (P-1)^{d-1}$ (see, e.g., [5]). Thus, there is a $\mathbf{g}^*\in G$ such that

$$\begin{aligned}\left(\sum_{\mathbf{mg}^*\equiv 0,\, \mathbf{m}\in\Omega_P}^{*} (\hat{K}_{\mathbf{a}}(\mathbf{m}))^{1/r}\right)^{r} &\leqslant \left(P^{-1} \sum_{\mathbf{m}\in\Omega_P} \prod_{s=1}^{v} (\underline{m}_s)^{-1} \prod_{s=v+1}^{d} (\underline{m}_s)^{-a_s/r}\right)^{r}\\ &\leqslant CP^{-r}(\log P)^{rv}.\end{aligned} \tag{10}$$

Combination of (8), (9), and (10) gives us (7). ∎

We can combine the two previous propositions to arrive at

THEOREM 3. *Suppose that* $1\leqslant p\leqslant 2$. *Then, there exists* $\mathbf{g}^*\in G$ *such that*

$$\sup_{f\in W_p^{\mathbf{a}}} \|f-\mathrm{sk}(f, \Delta_P^{\mathbf{g}^*})\|_{\infty} \leqslant CP^{-r+1/p}(\log P)^{rv-1/p}.$$

Proof. Setting $Q=P(\log P)^{-v}$ we see, from Proposition 1, that

$$\begin{aligned}\left(\sum_{\mathbf{z}\notin\Gamma_Q} |c_{\mathbf{z}}(\mathbf{w})|^p\right)^{1/p} &\leqslant CP^{-r+1/p}(\log P)^{(v-1)/p+v(r-1/p)}\\ &= CP^{-r+1/p}(\log P)^{vr-1/p}.\end{aligned}$$

For sufficiently large P the conditions of Proposition 2 are satisfied, so that

$$\left(\sum_{\mathbf{z}\in\Gamma_Q} |c_{\mathbf{z}}(\mathbf{w})|^p\right)^{1/p} \leqslant CP^{-r+1/p}(\log P)^{r\nu-\nu/p+(\nu-1)/p}$$

$$= CP^{-r+1/p}(\log P)^{\nu r-1/p}.$$

Substituting these results into (5) gives the required result. ∎

In order to extend these results to a wider range of p and different norms, we require the following duality result:

PROPOSITION 3. *Let* $1/p+1/p'=1$ *and* $1/q+1/q'=1$. *Then, for any* $\mathbf{g}\in G$,

$$\sup_{f\in W^{\mathbf{a}}_{q'}} \|f-\mathrm{sk}(f,\Delta^{\mathbf{g}}_P)\|_{p'} \leqslant C\left(\sup_{f\in W^{\mathbf{a}}_p} \|f-\mathrm{sk}(f,\Delta^{\mathbf{g}}_P)\|_q+\sum_{\mathbf{zg}\equiv 0} \hat{K}_{\mathbf{a}}(\mathbf{z})\right),$$

where C *is a constant.*

Proof. First we write $f=K_{\mathbf{a}}*\phi+c$ for some $\phi\in U_{q'}$ and $c\in\mathbb{R}$. Because $\int K_{\mathbf{a}}=0$ we may assume that $\int\phi=0$ as long as we admit functions whose norm is at most 2 rather than 1. So we consider $\phi\in 2U^0_{q'}$, where U^0_σ is the set of elements of U_σ with zero integral. Also, by Lemma 2, $\mathrm{sk}(f,\Delta^{\mathbf{g}}_P)=f$ if f is a constant. Hence,

$$\sup_{f\in W^{\mathbf{a}}_{q'}} \|f-\mathrm{sk}(f,\Delta^{\mathbf{g}}_P)\|_{p'}$$

$$\leqslant \sup_{\phi\in 2U^0_{q'}} \|f-\mathrm{sk}(f,\Delta^{\mathbf{g}}_P)\|_{p'}$$

$$= \sup_{\phi\in 2U^0_{q'}} \|K_{\mathbf{a}}*\phi-\mathrm{sk}(K_{\mathbf{a}}*\phi,\Delta^{\mathbf{g}}_P)\|_{p'}$$

$$= \sup_{\phi\in 2U^0_{q'}}\ \sup_{h\in U_p} \left|\int h(\mathbf{w})((K_{\mathbf{a}}*\phi)(\mathbf{w})-[\mathrm{sk}(K_{\mathbf{a}}*\phi,\Delta^{\mathbf{g}}_P)](\mathbf{w}))\,d\mathbf{w}\right|, \tag{11}$$

by the converse of Hölder's inequality.

We have, because $K_{\mathbf{a}}$ is even,

$$\int h(\mathbf{w})(K_{\mathbf{a}}*\phi)(\mathbf{w})\,d\mathbf{w}=\int (K_{\mathbf{a}}*h)(\mathbf{w})\,\phi(\mathbf{w})\,d\mathbf{w}. \tag{12}$$

Now we examine the term

$$\int h(\mathbf{w})[\operatorname{sk}(K_{\mathbf{a}} * \phi, \Delta_P^{\mathbf{g}})](\mathbf{w})\, d\mathbf{w}$$

$$= \int h(\mathbf{w}) \left\{ \sum_{i=1}^{P} (K_{\mathbf{a}} * \phi)(\mathbf{w}_i)\, \widetilde{\operatorname{sk}}_P^{\mathbf{g}}(\mathbf{w} - \mathbf{w}_k) \right\} d\mathbf{w}$$

$$= \sum_{i=1}^{P} (K_{\mathbf{a}} * \phi)(\mathbf{w}_i)(\widetilde{\operatorname{sk}}_P^{\mathbf{g}} * h)(\mathbf{w}_j),$$

using the fact that $\widetilde{\operatorname{sk}}_P^{\mathbf{g}}$ is even (Lemma 3(a)). But, by Lemma 3(b)(i),

$$\widetilde{\operatorname{sk}}_P^{\mathbf{g}}(\mathbf{w}) = \frac{1}{P} + \sum_{j=1}^{P} c_j K_{\mathbf{a}}(\mathbf{w} - \mathbf{w}_j).$$

Hence,

$$\int h(\mathbf{w})[\operatorname{sk}(K_{\mathbf{a}} * \phi, \Delta_P^{\mathbf{g}})](\mathbf{w})\, d\mathbf{w}$$

$$= \sum_{i=1}^{P} (K_{\mathbf{a}} * \phi)(\mathbf{w}_i) \left\{ \frac{1}{P} \int h(\mathbf{w})\, d\mathbf{w} + \sum_{j=1}^{P} c_j (K_{\mathbf{a}} * h)(\mathbf{w}_i + \mathbf{w}_j) \right\}$$

$$= \frac{1}{P} \int h(\mathbf{w})\, d\mathbf{w} \sum_{i=1}^{P} (K_{\mathbf{a}} * \phi)(\mathbf{w}_i)$$

$$+ \sum_{i=1}^{P} \sum_{j=1}^{P} c_j (K_{\mathbf{a}} * \phi)(\mathbf{w}_i)(K_{\mathbf{a}} * h)(\mathbf{w}_{i+j}). \tag{13}$$

Reordering the sum over j we get, in mind of the fact that $c_{(i-j)(\operatorname{mod} P)} = c_{(j-i)(\operatorname{mod} P)}$ (Lemma 3(b)(ii)),

$$\sum_{i=1}^{P} \sum_{j=1}^{P} c_j (K_{\mathbf{a}} * \phi)(\mathbf{w}_i)(K_{\mathbf{a}} * h)(\mathbf{w}_{i+j})$$

$$= \sum_{i=1}^{P} \sum_{j=1}^{P} c_{(j-i)(\operatorname{mod} P)} (K_{\mathbf{a}} * \phi)(\mathbf{w}_i)(K_{\mathbf{a}} * h)(\mathbf{w}_j)$$

$$= \sum_{i=1}^{P} \sum_{j=1}^{P} c_{(i-j)(\operatorname{mod} P)} (K_{\mathbf{a}} * \phi)(\mathbf{w}_i)(K_{\mathbf{a}} * h)(\mathbf{w}_j)$$

$$= \sum_{j=1}^{P} (K_{\mathbf{a}} * h)(\mathbf{w}_j) \sum_{j=1}^{P} c_i (K_{\mathbf{a}} * \phi)(\mathbf{w}_i + \mathbf{w}_j)$$

$$= \int \phi(\mathbf{w})[\operatorname{sk}(K_{\mathbf{a}} * h, \Delta_P^{\mathbf{g}})](\mathbf{w})\, d\mathbf{w} - \frac{1}{P} \int \phi(\mathbf{w})\, d\mathbf{w} \sum_{i=1}^{P} (K_{\mathbf{a}} * h)(\mathbf{w}_i),$$

where, to produce the last line, we have simply reversed the argument used to produce (13). However, $\phi \in 2U^0_{q'}$, so that, substituting the last equation and (12) into (11) we see that

$$\begin{aligned}
&\sup_{\phi \in 2U^0_{q'}} \sup_{h \in U_p} \left| \int h(\mathbf{w})((K_{\mathbf{a}} * \phi)(\mathbf{w}) - [\mathrm{sk}(K_{\mathbf{a}} * \phi, \Delta^{\mathbf{g}}_P)](\mathbf{w}))\, d\mathbf{w} \right| \\
&\quad = \sup_{h \in U_p} \sup_{\phi \in 2U^0_{q'}} \left| \int \phi(\mathbf{w})((K_{\mathbf{a}} * h)(\mathbf{w}) - [\mathrm{sk}(K_{\mathbf{a}} * h, \Delta^{\mathbf{g}}_P)](\mathbf{w}))\, d\mathbf{w} \right. \\
&\qquad \left. + \frac{1}{P} \int h(\mathbf{w})\, d\mathbf{w} \sum_{i=1}^{P} (K_{\mathbf{a}} * \phi)(\mathbf{w}_i) \right| \\
&\quad \leqslant 2 \sup_{h \in U_p} \| K_{\mathbf{a}} * h - \mathrm{sk}(K_{\mathbf{a}} * h, \Delta^{\mathbf{g}}_P) \|_q + \sup_{\phi \in 2U^0_{q'}} \left| \frac{1}{P} \sum_{i=1}^{P} (K_{\mathbf{a}} * \phi)(\mathbf{w}_i) \right|,
\end{aligned}$$

because $h \in U_p$. The result follows on observing that

$$\begin{aligned}
\frac{1}{P} \sum_{j=1}^{P} (K_{\mathbf{a}} * \phi)(\mathbf{w}_j) &= \frac{1}{P} \sum_{j=1}^{P} \sum_{\mathbf{z} \in \mathbb{Z}^d}^{*} \hat{K}_{\mathbf{a}}(\mathbf{z})\, \hat{\phi}(\mathbf{z})\, e^{i\mathbf{z}\mathbf{w}_j} \\
&= \sum_{\mathbf{z} \in \mathbb{Z}^d}^{*} \hat{K}_{\mathbf{a}}(\mathbf{z})\, \hat{\phi}(\mathbf{z}) \left\{ \frac{1}{P} \sum_{j=1}^{P} e^{i\mathbf{z}\mathbf{w}_j} \right\} \\
&= \sum_{\mathbf{z}\mathbf{g} \equiv 0}^{*} \hat{K}_{\mathbf{a}}(\mathbf{z})\, \hat{\phi}(\mathbf{z}) \\
&\leqslant C \sum_{\mathbf{z}\mathbf{g} \equiv 0}^{*} \hat{K}_{\mathbf{a}}(\mathbf{z}),
\end{aligned}$$

because $\hat{\phi}(\mathbf{z})$ are uniformly bounded, and $\hat{K}_{\mathbf{a}}(\mathbf{z}) > 0$, $\mathbf{z} \in \mathbb{Z}^d$. ∎

COROLLARY 1. *Suppose that* $1 \leqslant p \leqslant 2$. *Then, there exists* $\mathbf{g}^* \in G$ *such that*

$$\sup_{f \in W^{\mathbf{a}}_1} \| f - \mathrm{sk}(f, \Delta^{\mathbf{g}^*}_P) \|_{p'} \leqslant C P^{-r + 1/p} (\log P)^{rv - 1/p}.$$

Proof. The result follows directly from Theorem 3 and Proposition 3 on observing that

$$\sum_{\mathbf{z}\mathbf{g} \equiv 0}^{*} \hat{K}_{\mathbf{a}}(\mathbf{z}) \leqslant C P^{-r + 1/p} (\log P)^{rv - 1/p},$$

from (7), since the quantity on the left above is $S_{\mathbf{0}}$. ∎

An application of the Riesz–Thorin interpolation theorem between the pairs (p, ∞) (Theorem 3) and $(1, p')$ (Corollary 1) leads to the final result of the paper.

COROLLARY 2. *Suppose that* $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$, *with* $p^{-1} - q^{-1} \geqslant 1/2$. *Then, there exists* $\mathbf{g}^* \in G$ *such that*

$$\sup_{f \in W_p^{\mathbf{a}}} \| f - \mathrm{sk}(f, \Delta_P^{\mathbf{g}^*}) \|_q \leqslant C P^{-r + 1/p - 1/q} (\log P)^{rv - 1/p + 1/q}.$$

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